Efficiency and performances of finite difference schemes in the solution of saint Venant’s equation
Kalita H. M\textsuperscript{1}, Sarma A. K\textsuperscript{2}

1- Research Scholar, Department of Civil Engineering, Indian Institute of Technology, Guwahati, India
2- Professor, Department of Civil Engineering, Indian Institute of Technology, Guwahati, India
hriday@iitg.ernet.in
doi:10.6088/ijcser.00202030022

ABSTRACT

The computation of unsteady free-surface flows is required to estimate the arrival time and height of a flood wave at any downstream location in a river flow. The governing equations solved in this regards are the St. Venant equations. These equations are highly non-linear partial differential equation and closed form solutions are not available except in very simplified one-dimensional case. The computer revolution in twentieth century made a new era where numeric methods can be utilized effectively to solve nonlinear partial differential equations. Though several numerical schemes have been developed by various investigators for solving these equations, the choosing of efficient scheme regarding its efficiency, accuracy and computational effort has always been remaining as an important topic from very beginning. In this paper mainly results of two different schemes, Lax diffusive scheme and Beam and warming scheme are presented. Validations of the models were achieved by comparing their results with results of MIKE21C modeling tool. The results in terms of surface profiles at different times and velocity vectors for both the schemes when plotted, it was observed that both the schemes lead to same results. But in case of the Beam and Warming scheme, it was observed that it takes a very large computational time for simulation.

Keywords: Unsteady flow, Lax diffusive scheme, Beam and Warming scheme, Thomas algorithm, MIKE21C.

1. Introduction

The computation of unsteady free-surface flows is required to estimate the arrival time and height of a flood waves at any downstream location in a river flow. Though one dimensional (1D) numerical model have been used extensively in this regards, in many situations, however, one-dimensional solution is not sufficient to describe the actual flow scenario. For example, to study the changes in different flow variables at different places of a same cross-section in a river, the computation of two dimensional (2D) unsteady free surface flows becomes necessary. The computation of two-dimensional unsteady flow is more complicated than the computation of one-dimensional flows due to need for efficient solver routines and the inclusions of proper boundary conditions.

The shallow-water equations, also referred to as the St. Venant’s equations, describe two-dimensional unsteady free-surface flows, which are derived assuming hydrostatic pressure distribution. They are nonlinear first-order, hyperbolic partial differential equations for which closed-form solutions are not available, except in very simplified one-dimensional cases. Therefore, these equations are solved numerically. In this paper detail of the study done on
comparison of explicit and implicit schemes for simulating the unsteady flow in a channel and the results obtained from the study are presented.

2. Methodologies

A review of literature reveals that, explicit and implicit finite-difference methods have been used by a number of investigators (Fennema and Chaudhry, 1990; Ramesh et al., 2000; Patricia and Raimundo, 2005) for solving these equations. Some of them (Jingxiang and Charles, 1985; Chintu, 1998) used the method of characteristics to solve the unsteady flow equation in a channel flow. Whereas some of them (Khan, 2000; Schwanenberg and Harms, 2004; Weiming, 2004; Yong, 2010) have also applied finite-element and finite volume techniques in this regards. Akbari et al., 2010 compared the explicit and implicit finite difference scheme for solution of the St. Venant’s equation in 1D and obtained same results for both the schemes. From the survey of literature it appears that there is a scope of doing comparative study between explicit and implicit finite difference schemes for solving the St. Venant’s equations in 2D for evaluating the performances of the schemes.

2.1 Governing Differential Equations

St. Venant’s Equations describing unsteady free-surface flows may be derived by applying the laws of conservation of mass and momentum and by assuming hydrostatic pressure distribution, small channel bottom slopes.

For prismatic channels having no lateral inflow and outflow, the St. Venant’s equations in 2D in matrix form (Fennema et al. 1990) are as follows,

\[
U_t + E_x + F_y + S = 0
\]  

(1)

Where, 

\[
U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad E = \begin{pmatrix} hu^2 + gh^2 / 2 \\ hu^2 + gh^2 / 2 \\ hv^2 + gh^2 / 2 \end{pmatrix}, \quad F = \begin{pmatrix} hu \\ hv \\ hv \end{pmatrix}, \quad S = \begin{pmatrix} -gh(S_{ox} - S_{tx}) \\ -gh(S_{oy} - S_{ty}) \\ 0 \end{pmatrix}
\]

Where, \( h \) is flow depth; \( u \) is flow velocity in x direction; \( v \) is the flow velocity in y direction; \( S_{ox} \) is channel bottom slope in x direction; \( S_{oy} \) is channel bottom slope in y direction; \( x \) is the distance along the channel length; \( n \) is Manning’s coefficient; \( t \) is the time; and \( g \) is the acceleration due to gravity.

2.2 Finite Difference Method

The governing equation (1) is a set of nonlinear first-order, hyperbolic partial differential equations for which closed-form solutions are not available, except in very simplified one-dimensional cases. Therefore, these equations are solved numerically. One of the common numerical methods applied for the solution of these types of equation is finite difference method. The finite difference techniques are based upon the approximations that permit replacing differential equations by finite difference equations. These finite difference
Efficiency and performances of finite difference schemes in the solution of saint Venant’s equation
Kalita H. M, Sarma A. K

approximations are algebraic in form, and the solutions are related to grid points. After that the equations are solved at each grid point with appropriate initial and boundary conditions to obtain the values of unknown variables.

2.3 Lax Diffusive Scheme

When a direct computation of the dependent variables can be made in terms of known quantities, the computation is said to be explicit. Lax diffusive finite difference scheme (Chaudhry, 2008) is an explicit scheme. It is first order accurate in time and second order accurate in space. Flow variables are known at time level k and their values are to be determined at time level (k+1). The finite difference equation written for this scheme is as follows,

\[ U^{k+1}_{i,j} = \frac{1}{2} \left( U^{k}_{i,j+1} - U^{k}_{i,j-1} \right) - \frac{\Delta t}{2\Delta x} \left( E^{k}_{i,j+1} - E^{k}_{i,j-1} \right) - \frac{\Delta t}{2\Delta y} \left( F^{k}_{i+1,j} - F^{k}_{i-1,j} \right) - \Delta t S^{*}_{i,j} \]  (2)

Where, \( \Delta x \) is the grid spacing in x direction, \( \Delta y \) is the grid spacing in y direction, \( \Delta t \) is the grid spacing in t direction. From equation (2), \( U \) is computed at unknown time step. And then the values of primitive variables are determined from the computed value of \( U \) at each step from the formula (3) as follows,

\[ h^{k+1} = h^{k+1}, \quad u^{k+1} = \frac{h u^{k+1}}{h^{k+1}}, \quad v^{k+1} = \frac{h v^{k+1}}{h^{k+1}} \]  (3)

2.4 Beam and Warming Scheme

When the dependent variables are defined by coupled sets of equations, and either a matrix or iterative technique is needed to obtain the solution, the numerical method is said to be implicit finite difference scheme. Beam and Warming scheme (Beam and Warming, 1976) is an Alternate Direction Implicit scheme which is second order accurate in time and space. The solution is advanced in a two-step (double sweep) sequence, where each step involves the solution of a block tridiagonal system. For the efficient solution of the block tridiagonal system, Thomas algorithm (Niyogi, 2009) was used. In the first sweep (x-direction), an intermediate solution \( U^* \) is obtained. This intermediate value is then used in the second sweep (y-direction) to obtain the solution \( U^{**} \). The equations solved in x-direction and y-direction respectively is as follows,

\[ \left[ I + \Delta t \frac{\theta}{1 + \gamma} \frac{\partial}{\partial x} A \right] U^* = -\Delta t \frac{1}{1 + \gamma} \left( \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + S \right)^k + \frac{\gamma}{1 + \gamma} \Delta t U^k \]  (4)

\[ \left[ I + \Delta t \frac{\theta}{1 + \gamma} \frac{\partial}{\partial y} B + Q \right]^k U^{**} = U^* \]  (5)

And the final increment in U matrix, i.e. \( \Delta U^{k+1} \) is then calculated as,

\[ \Delta U^{k+1} = U^k + U^{**} \]  (6)

In equations (3) and (4), \( I \) is identity matrix; \( A, B \) and \( Q \) are the Jacobians of matrices \( E, F \) and \( S \) respectively; \( \theta \) and \( \gamma \) are factors, values of which lead to different formulations of Beam and Warning scheme (Chaudhry, 2008). In this project \( \theta=0.5 \) and \( \gamma=0 \) (Trapezoidal formula) were used. From equation (6), the increment in the U matrix is calculated at each
time step and the primitive flow variables at each step are calculated from the formula as shown below,

\[ \begin{align*}
    h_{ij}^{k+1} &= h_{ij}^k + \Delta h_{ij}^{k+1}, \\
    u_{ij}^{k+1} &= \frac{uh_{ij}^{k} + \Delta uh_{ij}^{k+1}}{h_{ij}^{k+1}}, \\
    v_{ij}^{k+1} &= \frac{vh_{ij}^{k} + \Delta vh_{ij}^{k+1}}{h_{ij}^{k+1}}
\end{align*} \quad (7) \]

3. Application of the schemes

The above numerical schemes are then applied to study the unsteady flow behaviour in a prismatic trapezoidal channel by routing a hydrograph through the channel. The details of the tests done for both the schemes and their results are presented below.

3.1 Channel specification

Test channel used for both the simulation was 20 km in length and 1 km in top width. The bottom width of the channel was 0.2 km and the side slope was 1:63. The total length of the channel was divided into 40 grids with \( \Delta x = 500 \) meter, and the total width of the channel was divided into 5 grids with \( \Delta y = 200 \) meter. The roughness of the channel was introduced by applying manning’s \( n \). A classical value of 0.031 was applied over the whole computational domain for both the schemes. The longitudinal bed slope of the channel was assumed to be 1:10000.

3.2 Initial and Boundary condition

Values of \( h, u \) and \( v \) at the beginning of time were given at all the nodes as initial condition in both the schemes. At upstream a flow hydrograph as shown in figure 1 was used. At downstream all dependent variables were extrapolated from interior domain. The solid walls were simulated as no flux free slip boundary for both the schemes. The normal velocity component \( v \) was assumed to be zero at those lines. In the implicit scheme the remaining values at the solid boundaries were extrapolated from interior grids. But, in the explicit scheme reflective boundary condition technique (Fennema and Chaudhry, 1990) was used.

![Image showing upstream flow hydrograph](image)

**Figure 1:** Image showing upstream flow hydrograph

3.3 Stability of the models
Both the schemes applied here are time marching technique. So for stability they must obey Courant–Friedrichs–Lewy (CFL) condition. The CFL conditions for 2D flow are as follows,

$$\max \left( \frac{(u + \sqrt{gh}) \Delta t}{\Delta x} \right) \leq 1, \quad \max \left( \frac{(v + \sqrt{gh}) \Delta t}{\Delta y} \right) \leq 1$$

Thus, the computational time interval depends upon the spatial grid spacing, flow velocity, and celerity, which are functions of the flow depth. Since the flow depth and the flow velocity may change significantly during the computations, it may be necessary to reduce the size of computational time interval for stability.

### 3.4 Validation with MIKE21C model

MIKE 21C, developed by the Danish Hydraulics Institute, is a software package for simulating free-surface flows in two dimensions. The governing equations (Saint Venant’s equations) are solved in this modeling tool in curvilinear coordinate and by finite difference implicit scheme. A constant discharge was applied at upstream for all the models including MIKE21C, and a corresponding water stage was applied at downstream for the same. At a cross-section 10 km downstream from upstream boundary the velocities and water depths were compared for all the cases as shown in figure 2 and figure 3 respectively.

**Figure 2:** Image showing comparison of velocity

**Figure 3:** Image showing comparison of depth
4. Results and discussions

By using the above initial and boundary conditions the governing equations of free surface flow were solved by both the schemes. Figure 4 shows the water surface profiles along the channel after elapsing of 1900 seconds from beginning for both the schemes. And figure 5 shows the same after elapsing of 5000 seconds.

From the figures of surface elevations it has been observed that both explicit and implicit methods were leading to same surface elevations at different times of flow. Figure 6 and figure 7 shows the velocity vectors after elapsing of 900 seconds from beginning for the
Efficiency and performances of finite difference schemes in the solution of saint Venant’s equation
Kalita H. M, Sarma A. K

In International Journal of Civil and Structural Engineering
Volume 2 Issue 3 2012

explicit scheme and implicit scheme respectively. It was observed that both the models produce same flow velocities and directions.

![Figure 6: Image showing velocity vectors for Lax diffusive scheme](image1)

![Figure 7: Image showing velocity vectors for Beam and Warming scheme](image2)

Both the schemes applied here are stable, only if they satisfy CFL condition. In this way the value of $\Delta t$ in both the schemes became same as $\Delta x$ and $\Delta y$ are having same values in both the schemes. In the Lax diffusive scheme applied here the unknown variables are being calculated directly from the known variables, whereas in the Beam and Warming scheme a series of block tridiagonal matrices are needed to be solved to obtain the values of unknown variables. This leads to very larger time required for simulation in Beam and Warming scheme than that of the Lax diffusive scheme. The time required in Beam and Warming solution for the above problem was observed six times more than that of the time required in Lax diffusive solution.

5. Conclusion

Saint-Venant’s equations in two dimensions are solved for hypothetical flood routing problems in a trapezoidal channel through Lax diffusive explicit and Beam and Warming implicit numerical schemes. The following conclusions can be obtained from the study,
1. When validated with MIKE21C results it was observed that these models produce same results as that of MIKE21C.

2. After observing the surface elevation profiles at different times and the velocity vectors it was observed that there is a reasonably good matching in the model results.

3. Computational time required in implicit scheme was observed very high than that of the explicit scheme due to the lengthy procedures in the implicit scheme.

6. References

1. Akbari, G. and Firoozi, B., (2010), Implicit and Explicit Numerical Solution of Saint-Venant’s Equations for Simulating Flood Wave in Natural Rivers, 5th National Congress on Civil Engineering, May 4-6, Ferdowsi University of Mashhad, Mashhad, Iran.


9. Patricia, C. and Raimundo, S., (2005), Solution of Saint Venant’s equation to study flood in rivers, through numerical methods, Hydrology days, Department of Environmental and Hydraulics engineering, Federal university of Ceara.

