Determination of 3D Transformation Parameters for the Ghana Geodetic Reference Network using Ordinary Least Squares and Total Least Squares Techniques

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ABSTRACT

Ghana has been utilizing Global Navigation Satellite Systems (GNSS) technology for geodetic purposes. In order to apply the GNSS acquired data locally, there is the need to transform coordinates from the geocentric datum to the Ghana local geodetic datum (non-geocentric). This is usually carried out by applying least squares adjustment technique to a coordinate transformation model. The present study compared both the ordinary least squares and total least squares adjustment techniques in 3D coordinate transformation between the Ghana local geodetic datum and the global geocentric datum. These least squares techniques were applied to the Bursa-Wolf and Molodensky-Badekas model. The statistical analytical results revealed that the ordinary least squares and total least squares gave closely related transformation results. The main motivation of this work is to ascertain the extent of applicability and performance of ordinary least squares and total least squares for the first time in Ghana geodetic reference network. The reason is that, although the aforementioned least squares methods have been compared and reported in several studies, such kind of work is yet to be done in the Ghana geodetic reference network. Therefore, this study will help Land surveyors and other built environment practitioners in Ghana to know the efficiency of the two least squares methods and apply them accordingly.

Keywords: Total Least Squares, Ordinary Least squares, Bursa-Wolf Model, Molodensky-Badekas Model, Coordinate Transformation

1. Introduction

In the fields of geodesy and geospatial engineering, positions of points are measured with reference to the Earth surface to establish a well-defined coordinate systems and datums. To accomplish this task, control surveys (both horizontal and vertical) are used as a means of precisely establishing positions on the Earth surface. Previously, horizontal control networks for countries were established through conventional surveying observation techniques such as triangulation, trilateration, intersection, resection and astronomy (Constantin-Octavian, 2006; Dzidefo, 2011).

Currently, the advent of Global Navigation Satellite Systems (GNSS) has revolutionised the field of Geomatics as well as its related disciplines. As an example, geodetic and geospatial surveys in Ghana are now being dominated by GNSS due to its cost effectiveness, speed and
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accuracy (Dadzie et al., 2008) as compared to the traditional surveying methods. However, up until now, Ghana has not been able to fully utilize the enormous potential of GNSS due to some limitations. One of the basic contributing factors is that, the reference ellipsoid known as the War Office 1926 adopted for mapping and engineering projects in Ghana is non-geocentric, whereas the GNSS based on World Geodetic System 1984 (WGS84) is geocentric. This means that, the two datums (global and local) have different origin, size and shape, respectively. Consequently, for local application, there is the need to localise the GNSS acquired data into the local geodetic reference frame.

Localization of the GNSS acquired data is done by the use of coordinate transformation techniques. In Ghana, the similarity models of Bursa-Wolf (Bursa, 1962; Wolf, 1963), Molodensky-Badekas (Molodensky et al., 1962; Badekas, 1969), Abridged Molodensky (Molodensky et al., 1962) and Veis (Veis, 1960) have been the most commonly used approaches. This assertion is confirmed by the various coordinate transformation studies carried out in Ghana (see Ayer, 2008; Ayer and Fosu, 2008; Dzidefo, 2011; Ziggah et al., 2013a, 2013b; Kumi-Boateng and Ziggah, 2016; Anann et al., 2016). It is needed to note that the similarity models are dependent on a set of parameters to accomplish the transformation task. Because of redundant but independent observations, the parameters are usually determined by the least squares adjustment procedure.

The ordinary least squares (OLS) approach has been mostly used in Ghana to determine the transformation parameters to be applicable in the similarity models (Ayer, 2008; Ayer and Fosu, 2008; Dzidefo, 2011; Ziggah et al., 2013a, 2013b; Kumi-Boateng and Ziggah, 2016). This OLS technique considers error in only the observation vector. However, in surveying and mapping, all observed data (coordinates) may suffer from errors. This is usually ignored in the OLS technique. A procedure known as the total least squares (TLS) has been recommended in recent times as alternate adjustment method that incorporate errors in both coordinates. Mathematically, errors in the observation vector and design matrix are taken into account (Felus and Schaffrin, 2005).

This has led to studies applying and comparing the TLS to OLS for coordinate transformation in geodetic networks across different countries (Okwuashi and Eyoh, 2012; Mahboub, 2012; Acar et al., 2006a, 2006b; Akyilmaz, 2007; Felus and Schaffrin, 2005). These studies indicated that the TLS is computationally robust compared to OLS. It was also noticed that, in most cases, the TLS outperformed the OLS. It must be acknowledged here that in Ghana, the TLS has been used by Anann et al. (2016) to determine the transformation parameters using the Molodensky-Badekas model. However, the objective of their study was to propose a novel centroid computation technique that could produce better reliable results when applied to the Molodensky-Badekas model. Hence, only the TLS approach was used.

It is fair to state that, although the efficiency of the TLS and OLS in coordinate transformation has been successfully carried out in several countries, a work of this kind is yet to be done for the Ghana geodetic reference network. Therefore, the present authors were motivated to examine the performance of TLS and OLS for Ghana geodetic reference network. These least squares techniques (OLS and TLS) were applied on two similarity based models, namely, Bursa-Wolf and Molodensky-Badekas, respectively. The choice for these models is because they are the commonly used conformal models used by researchers in Ghana and other countries due to their simplicity in application and achievable accuracy. The assessment of Bursa-Wolf and Molodensky-Badekas model performance was carried out using the estimated horizontal positional error, standard deviation, maximum and minimum
errors, respectively. The results revealed that, for Ghana geodetic reference network, closely related results can be achieved by OLS and TLS.

2. Study area and resources used

This study was carried out in the Ghana geodetic reference network. It is a network of monuments whose coordinates date back to the colonial era of Captain Gordon Guggisberg, the Governor of the by then Gold Coast now Ghana (REF). The Accra datum based on the War Office 1926 ellipsoid is used in Ghana for all geospatial purposes. The War Office 1926 ellipsoidal properties include semi-major axis \(a = 6378299.99899832\) m, a semi-minor axis \(b = 6356751.68824042\) m, and flattening \(f = 1/296\) (Thomas et al., 2000; Ayer and Fosu, 2008).

With the increased patronage of GNSS, the Ghana Survey and Mapping Division of Lands Commission, embarked on the establishment of GNSS reference network based on the International Terrestrial Reference System (ITRS). This was carried out through the Land Administration Project (LAP) sponsored by the World Bank (Kotzev, 2013). The LAP has been divided into phases, the first phase (which is completed) covering the mid-southern half of Ghana is locally known as the golden triangle (Fig. 1). Three permanently operating reference stations have been established at the apexes of the triangle with nineteen second order stations well distributed spatially within the study area (Poku-Gyamfi and Schueler, 2008). The GPS obtained satellite coordinates was defined in the International Terrestrial Reference Frame 2005 (ITRF2005) specified at epoch 2007.39 (Kotzev, 2013).

![Figure 1: Study area](image)

Differential processing of the GPS receiver coordinates with the International GNSS Service (IGS) provided the resulting latitude, longitude and ellipsoidal height for the WGS84 (Dzidefo, 2011; Kotzev, 2013). The coordinates for the War Office 1926 reference frame established by classical surveying procedures were latitude, longitude and orthometric height. This information was obtained from the Ghana Survey records (1936). Figure 1 shows the points distribution of the data covering the golden triangle. In this study two sets of nineteen
common points (Table 1) from the LAP in War Office 1926 \((\phi, \lambda, h)_F\) and WGS84 \((\phi, \lambda, h)_W\) were used in the coordinate transformation process. Here, \((\phi, \lambda, h)\) is the geodetic latitude, geodetic longitude and ellipsoidal height, respectively.

**Table 1:** Common point coordinates in War Office 1926 and WGS84 reference frame

<table>
<thead>
<tr>
<th>PT ID</th>
<th>(\lambda_F)</th>
<th>(\phi_F)</th>
<th>(h_F)</th>
<th>(\lambda_W)</th>
<th>(\phi_W)</th>
<th>(h_W)</th>
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<tbody>
<tr>
<td>P1</td>
<td>-0.12229</td>
<td>5.936247</td>
<td>525.5954</td>
<td>-0.122</td>
<td>5.939034</td>
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<td>P2</td>
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<td>-0.7297</td>
<td>5.943107</td>
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<tr>
<td>P3</td>
<td>-0.76584</td>
<td>6.573035</td>
<td>780.2024</td>
<td>-0.76557</td>
<td>6.575797</td>
<td>782.2084</td>
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<tr>
<td>P4</td>
<td>-0.74918</td>
<td>6.125418</td>
<td>327.4004</td>
<td>-0.7489</td>
<td>6.128197</td>
<td>327.0218</td>
</tr>
<tr>
<td>P5</td>
<td>-1.16493</td>
<td>6.568592</td>
<td>613.9824</td>
<td>-1.16466</td>
<td>6.571357</td>
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</tr>
<tr>
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<td>7.233129</td>
<td>530.9264</td>
<td>-1.63047</td>
<td>7.235861</td>
<td>536.0062</td>
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<tr>
<td>P7</td>
<td>-2.01701</td>
<td>6.909735</td>
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<td>6.989461</td>
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<td>-1.44534</td>
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<td>88.3693</td>
<td>0.734406</td>
<td>5.282744</td>
<td>83.4515</td>
</tr>
<tr>
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<td>6.051007</td>
<td>438.7549</td>
<td>1.286211</td>
<td>6.053792</td>
<td>437.699</td>
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<tr>
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<td>6.482004</td>
<td>642.6334</td>
<td>1.925156</td>
<td>6.484775</td>
<td>643.5756</td>
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<td>P13</td>
<td>1.412161</td>
<td>6.554147</td>
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<td>1.411897</td>
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<td>82.06617</td>
<td>-0.42356</td>
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<td>-1.69483</td>
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<td>1.966153</td>
<td>5.849618</td>
<td>399.3477</td>
</tr>
</tbody>
</table>

3. Methods

3.1 Abridged Molodensky Transformation Model

As stated earlier, the War Office 1926 reference frame involved data in geodetic latitude, geodetic longitude and orthometric height. Hence, converting ellipsoidal coordinates to cartesian coordinates require the need to determine ellipsoidal height for the War Office 1926 ellipsoid. To carry out this task, the Abridged Molodensky transformation model was applied. Equations 1, 2 and 3 (NIMA, 1997) is the mathematical representation of the model.

\[
\Delta \phi = \frac{1}{\rho \sin \lambda} \left( -\Delta X \sin \phi \cos \lambda - \Delta Y \sin \phi \sin \lambda + \Delta Z \cos \phi + (a.\Delta f + f.\Delta a) \sin 2\phi \right) \tag{1}
\]
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\[ \Delta \lambda = \frac{1}{N \cos \phi \sin \lambda} (-\Delta X \sin \lambda + \Delta Y \cos \phi) \] \hspace{1cm} [2]

\[ \Delta h = \Delta X \cos \phi \cos \lambda + \Delta Y \cos \phi \sin \lambda + \Delta Z \sin \phi + (a \Delta f + f \Delta a) \sin^2 \phi - \Delta a \] \hspace{1cm} [3]

where \( \rho \) is defined in Eq. (4) as

\[ \rho = \frac{a(1-e^2)}{3(1-e^2 \sin^2 \phi)} \]. \hspace{1cm} [4]

Here, \( \rho \) is the radius of curvature in the meridian, \((\Delta \phi, \Delta \lambda, \Delta h)\) is the set of corrections to transform \((\phi, \lambda, h)_W\) to \((\phi, \lambda, h)_F\) and \((\Delta X, \Delta Y, \Delta Z)\) is the set of corrections to transform \((X, Y, Z)_W\) to \((X, Y, Z)_F\). The \( \Delta h \) (Equation 3) results were then used to compute the War Office 1926 ellipsoidal height through Eq. (5) given by

\[ h_F = h_W - \Delta h \] \hspace{1cm} [5]

where \( h_F \) is the War Office 1926 ellipsoidal height and \( h_W \) is the WGS84 ellipsoidal height.

Having determined the ellipsoidal height for the War Office 1926, Eqs. (6), (7) and (8) (Heiskanen and Moritz, 1967) were then used to convert \((\phi, \lambda, h)_F\) and \((\phi, \lambda, h)_W\) to cartesian coordinate coordinates. The obtained cartesian coordinates are denoted in this study as \((X, Y, Z)_F\) and \((X, Y, Z)_W\) for the two reference frames.

\[ X = (M + h) \cos \phi \cos \lambda \] \hspace{1cm} [6]

\[ Y = (M + h) \cos \phi \sin \lambda \] \hspace{1cm} [7]

\[ Z = \left[ M \left(1 - e^2\right) + h \right] \sin \phi \] \hspace{1cm} [8]

where \( M = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \) is the radius of curvature in the prime vertical, \( e^2 = 2f - f^2 \) is the first eccentricity, \( f \) is the flattening of the reference ellipsoid and \((\phi, \lambda, h)\) is the set of geodetic coordinates.

### 3.2 Bursa-Wolf Model

The Bursa-Wolf Model (Bursa, 1962; Wolf, 1963) relates two rectangular coordinate systems through Eq. (9) as

\[
\begin{bmatrix}
X_F \\
Y_F \\
Z_F
\end{bmatrix}
= 
\begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
+ \eta R(\alpha_1, \alpha_2, \alpha_3)
\begin{bmatrix}
X_W \\
Y_W \\
Z_W
\end{bmatrix}
\] \hspace{1cm} [9]
where $X_F$, $Y_F$ and $Z_F$ are the respective War Office 1926 reference frame coordinates;

$X_W$, $Y_W$ and $Z_W$ are the respective WGS 84 reference frame coordinates;

$T_X$, $T_Y$ and $T_Z$ are the respective translations along $x$-, $y$- and $z$-axes;

$\eta$ is the scale factor;

$R$ is the total rotational matrix (product of the rotation angles) given by Eq. (10) as

$$
R = \begin{bmatrix}
\cos \alpha_3 & \sin \alpha_3 & 0 \\
-\sin \alpha_3 & \cos \alpha_3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \alpha_2 & 0 & -\sin \alpha_2 \\
0 & 1 & 0 \\
\sin \alpha_2 & 0 & \cos \alpha_2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha_1 & \sin \alpha_1 \\
0 & -\sin \alpha_1 & \cos \alpha_1
\end{bmatrix}
$$

[10]

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are the respective rotation angles around $x$-, $y$- and $z$-axes. Expanding Eq. (9) gives Eq. (11) given as

$$
\begin{bmatrix}
X_W \\
Y_W \\
Z_W
\end{bmatrix}
= \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
+ \begin{bmatrix}
1 + \eta & R_Z & -R_Y \\
-R_Z & 1 + \eta & R_X \\
R_Y & -R_X & 1 + \eta
\end{bmatrix}
\begin{bmatrix}
X_F \\
Y_F \\
Z_F
\end{bmatrix}
$$

[11]

This study applied the method of least squares to determine the unknown transformation parameters ($X$) (Eq. (12)) by using the design matrix ($A$) (Eq. (13)) and the observation vector ($L$) (Eq. (14)).

$$
X = 
\begin{bmatrix}
T_X \\
T_Y \\
T_Z \\
R_X \\
R_Y \\
R_Z \\
\eta
\end{bmatrix}
$$

[12]

$$
A = 
\begin{bmatrix}
1 & 0 & 0 & 0 & -Z_F & Y_F & X_F \\
0 & 1 & 0 & Z_F & 0 & -X_F & Y_F \\
0 & 0 & 1 & -Y_F & X_F & 0 & Z_F
\end{bmatrix}
$$

[13]

$$
L = 
\begin{bmatrix}
X_W - X_F \\
Y_W - Y_F \\
Z_W - Z_F
\end{bmatrix}
$$

[14]

### 3.3 Molodensky-Badekas Model

The Molodensky-Badekas model (Molodensky et al., 1962; Badekas, 1969) relating two rectangular coordinate system is expressed in Eq. (15) as
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\[
\begin{bmatrix}
X_W \\
Y_W \\
Z_W
\end{bmatrix}
= \begin{bmatrix}
X_C \\
Y_C \\
Z_C
\end{bmatrix} + \begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix} + \eta R(\alpha_1, \alpha_2, \alpha_3) \cdot \begin{bmatrix}
X_F - X_C \\
Y_F - Y_C \\
Z_F - Z_C
\end{bmatrix}
\] \tag{15}

where \(X_C, Y_C\) and \(Z_C\) are the respective centroids of points in the War Office 1926 reference frame. With \(n\) as the number of observation points, the centroids are given by Eq. (16) as

\[
X_C = \frac{1}{n} \sum_{i=1}^{n} X_{F_i}, \quad Y_C = \frac{1}{n} \sum_{i=1}^{n} Y_{F_i} \quad \text{and} \quad Z_C = \frac{1}{n} \sum_{i=1}^{n} Z_{F_i}.
\] \tag{16}

Equation 15 was reduced into design matrix (A) (Eq. (17)), observation matrix (L) (Eq. (18)) and vector of unknown parameters (X) (Eq. (19)).

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & -(Z_F - Z_C) & (Y_F - Y_C) & (X_F - X_C) \\
0 & 1 & 0 & (Z_F - Z_C) & 0 & -(X_F - X_C) & (Y_F - Y_C) \\
0 & 0 & 1 & -(Y_F - Y_C) & (X_F - X_C) & 0 & (Z_F - Z_C)
\end{bmatrix}
\] \tag{17}

\[
L = \begin{bmatrix}
X_W - X_F - (X_F - X_C) \\
Y_W - Y_F - (Y_F - Y_C) \\
Z_W - Z_F - (Z_F - Z_C)
\end{bmatrix}
\] \tag{18}

\[
X = \begin{bmatrix}
T_X \\
T_Y \\
T_Z \\
R_X \\
R_Y \\
R_Z \\
\eta
\end{bmatrix}
\] \tag{19}

3.4 Ordinary Least Squares

A system of over determined equations is solved through OLS that takes the form of Eq. (20).

\[
AX = L + V_L
\] \tag{20}

The matrix of unknown parameters is estimated through Eq. (21) (a unit weight was assumed).

\[
X = \left( A^T A \right)^{-1} A^T L
\] \tag{21}

The error vector \(V_L\) associated with the OLS is given by Eq. (22) as

\[
V_L = AX - L
\] \tag{22}
3.5 Total Least Squares

Golub and Van Loan (1980) introduced TLS as a method of treating an over determined system of linear equations by solving for the unknown parameters, $\hat{X}$ in Eq. (23) through the form

$$L + V_L = (A + V_A)\hat{X}, \quad rank(A) = m < n,$$

where $V_L$ and $V_A$ is the vector of errors in the observation and the data matrix. Both $V_L$ and $V_A$ are assumed to have independent and identical distributed rows with zero mean and equal variance (Felus and Schaffrin, 2005; Akyilmaz, 2007).

The TLS method is an iterative algorithm that minimises the errors in Eq. (23) through a minimising matrix $[\hat{A}, \hat{L}]$; the iteration continues until any $\hat{X}$ that satisfies $\hat{A}\hat{X} = \hat{L}$ becomes the TLS solution (Golub and Van Loan, 1980; Akyilmaz, 2007; Yanmin et al., 2011; Okwuashi and Eyoh, 2012).

The Singular Value Decomposition (SVD) of the matrix $[A, L]$ was used in solving the TLS problem. SVD is used to represent $[A, L]$ through Eq. (24) as

$$[A, L] = USV^T$$

where $U = [U_1, U_2, U_1 = [u_1, \ldots, u_m], U_2 = [u_{m+1}, \ldots, u_n], U^T U = I_n$ and $u_i \in R^n$

$V = [v_1, \ldots, v_m, v_{m+1}], V^T V = I_{m+1}$ and $v_i \in R^{m+1}$

$$S = diag(\sigma_1, \ldots, \sigma_m, \sigma_{m+1}), S \in R^{n(m+1)}$$

Through the SVD, the solution for the TLS problem is finally given by Eq. (25) as

$$[\hat{X}, -I] = \frac{-1}{V_{m+1,m+1}}V_{m+1}.$$

If $V_{m+1,m+1} \neq 0$, then $\hat{L} = \hat{A}\hat{X} = -I/(V_{m+1,m+1})\hat{A}[v_1, \ldots, v_{m+1}, \ldots, v_{m,m+1}]^T$ which belongs to the column space of $\hat{A}$, so $\hat{X}$ solves the basic TLS problem (Okwuashi and Eyoh, 2012).

The corresponding TLS correction is expressed in Eq. (26) as

$$[A\hat{A}, \Delta L] = [A, L] - [\hat{A} - \hat{L}].$$

3.6 Accuracy Assessment

The assessment of the least squares methods was done by estimating the horizontal positional accuracies. This was achieved through the use of horizontal positional error (Eq. (27)), error standard deviation (Eq. (28)), maximum and minimum error, respectively.

$$HE = \sqrt{(E_d - E_p)^2 + (N_d - N_p)^2} = \sqrt{\Delta E^2 + \Delta N^2}$$
\[ SD = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (e - \bar{e})^2} \]  

Here, \( n \) is the number of observation, \( E_d \) and \( N_p \) are the measured and estimated Easting and Northing coordinates. \( e \) denote the residuals and \( \bar{e} \) is the mean of the residuals.

4. Results and discussion

The transformation parameters and their corresponding standard deviations obtained through OLS and TLS for the Bursa-Wolf and Molodensky-Badekas as models are respectively shown in Table 2.

**Table 2:** Transformation Parameter Estimates from Bursa-Wolf and Molodensky-Badekas models

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Bursa-Wolf OLS</th>
<th>Bursa-Wolf TLS</th>
<th>Molodensky-Badekas OLS</th>
<th>Molodensky-Badekas TLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_x (m) )</td>
<td>-151.18907±10.17137</td>
<td>-159.17764±10.96305</td>
<td>-196.62110±0.136037</td>
<td>-196.62237±0.13604</td>
</tr>
<tr>
<td>( T_y (m) )</td>
<td>31.59312±16.91510</td>
<td>57.13318±19.16235</td>
<td>322.34374±136038</td>
<td>322.34581±13604</td>
</tr>
<tr>
<td>( T_z (m) )</td>
<td>327.17669±16.87422</td>
<td>366.88082±19.11405</td>
<td>322.34374±136038</td>
<td>322.34581±13604</td>
</tr>
<tr>
<td>( R_x (\text{sec}) )</td>
<td>0.4451±1.70196E-06</td>
<td>0.4451±1.70196E-06</td>
<td>0.4451±1.70196E-06</td>
<td>0.4451±1.70196E-06</td>
</tr>
<tr>
<td>( R_y (\text{sec}) )</td>
<td>-0.0058±2.65214E-06</td>
<td>-0.0058±2.65214E-06</td>
<td>-0.0058±2.65214E-06</td>
<td>-0.0058±2.65214E-06</td>
</tr>
<tr>
<td>( R_z (\text{sec}) )</td>
<td>0.02199±2.98939E-06</td>
<td>0.02199±2.98939E-06</td>
<td>0.02199±2.98939E-06</td>
<td>0.02199±2.98939E-06</td>
</tr>
<tr>
<td>( \eta ) (ppm)</td>
<td>-7.16775±1.58270E-06</td>
<td>-7.16775±1.58270E-06</td>
<td>-7.16775±1.58270E-06</td>
<td>-7.16775±1.58270E-06</td>
</tr>
</tbody>
</table>

The standard deviations give an indication on the precision of the transformed coordinates produced from the model. Thus how well the transformed WGS84 coordinates will fit with the known War Office 1926 system. From Table 2, it can be seen that the Bursa-Wolf and Molodensky-Badekas models yielded identical rotational and scale factor results. This buttress the mathematical analogy presented in Rapp (1993) that the Bursa-Wolf and Molodensky model differ only in their estimated translation parameters. This difference has been attributed to the different ways the scale factor parameter (\( \eta \)) is implemented in the two models. That is, in the case of the Bursa-Wolf model, \( \eta \) is applied to all the position vectors, while \( \eta \) is applied to the coordinate differences in the Molodensky-Badekas model. This can easily be identified in Eqs. (9) and (15), respectively.

As a means of assessing the strength of the least squares techniques applied to Bursa-Wolf and Molodensky-Badekas model, the horizontal positional errors for each coordinates were estimated. Table 3 presents the computed horizontal positional errors for the data points.

**Table 3:** Horizontal positional error of the least squares methods

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Bursa-Wolf OLS (m)</th>
<th>Bursa-Wolf TLS (m)</th>
<th>Molodensky-Badekas OLS (m)</th>
<th>Molodensky-Badekas TLS (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.168 )</td>
<td>1.092</td>
<td>1.168</td>
<td>1.170</td>
<td></td>
</tr>
<tr>
<td>( 0.865 )</td>
<td>0.808</td>
<td>0.865</td>
<td>0.867</td>
<td></td>
</tr>
<tr>
<td>( 1.574 )</td>
<td>1.632</td>
<td>1.574</td>
<td>1.572</td>
<td></td>
</tr>
<tr>
<td>( 1.806 )</td>
<td>1.738</td>
<td>1.806</td>
<td>1.808</td>
<td></td>
</tr>
<tr>
<td>( 1.128 )</td>
<td>1.095</td>
<td>1.128</td>
<td>1.126</td>
<td></td>
</tr>
<tr>
<td>( 0.882 )</td>
<td>0.876</td>
<td>0.882</td>
<td>0.880</td>
<td></td>
</tr>
</tbody>
</table>
In Table 3, the extent at which the results produced by OLS and TLS deviated horizontally from the measured coordinates can be known. This further indicates the horizontal positional accuracy achieved when OLS and TLS was applied to transform coordinates from global datum (WGS84) to the local datum (War Office 1926).

A comparison of OLS and TLS applied to Bursa-Wolf model is shown in Fig. 2. The results presented in Fig. 2 clearly show that both OLS and TLS produced closely related transformation results. This can easily be seen from Fig. 2 where nearly equal error variation was observed between the interquartile range length of OLS and TLS, respectively. However, a closer look at Fig. 2 showed slightly lesser error variability for the TLS than OLS. This was also evident from the median error (Fig. 2) where the TLS was slightly better than OLS. It should be noted that on the whole, these error differences are very small and thus insignificant. This is because both the upper and lower whiskers (Fig. 2) of the least squares methods did not show any significant difference. This can also be known from outliers shown by the symbol ‘●’ in Fig. 2. The reason is that the OLS and TLS outliers are both concentrated in the same region. This can additionally be viewed from Fig. 3.

**Figure 2:** Horizontal positional error variability of the least squares methods (Bursa-Wolf model)
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Figure 3: Horizontal positional displacement of the least squares methods (Bursa-Wolf model)

For the Molodensky-Badekas model, the OLS and TLS produced identical error variability (Fig. 3). This can be seen from the interquartile error range for both OLS and TLS, respectively. Also, the outliers (●) for OLS and TLS are centred around the same error margin was also observed as shown in Fig. 3 The same was realised for the OLS and TLS error median (Fig. 3). These imply that identical horizontal positional accuracy could be achieved in the Ghana geodetic reference when OLS and TLS are applied to the Molodensky-Badekas model. Intuitive interpretation of Fig.4 confirms that assertion.

Figure 4: Horizontal positional error variability of the least squares methods (Molodensky-Badekas model)
Next, the OLS results for both Bursa-Wolf and Molodensky-Badekas model were analysed. Likewise, the TLS results for Bursa-Wolf and Molodensky-Badekas was done. The objective here is to evaluate and compare the results of OLS when applied to Bursa-Wolf and Molodensky-Badekas model. Similarly, the results of TLS for Bursa-Wolf and Molodensky were compared. This conceptual idea is further illustrated in Fig. 6. It is obvious from Fig. 6 that identical results were obtained when the OLS was applied to the Bursa-Wolf and Molodensky-Badekas. With regard to the TLS, it can be observed (Fig. 6) that Bursa-Wolf model attained slightly better horizontal error distribution than the Molodensky-Badekas model. The same can be noticed in terms of their outliers denoted as ‘.’ in Fig 6. This can be observed further from Fig. 7.

**Figure 5:** Horizontal positional displacement of the least squares methods (Molodensky-Badekas model)

**Figure 6:** Comparison of Horizontal positional error variability of Bursa-Wolf and Molodensky-Badekas. BW is Bursa-Wolf and MB is Molodensky-Badekas.
Figure 7: Horizontal positional shift of the Bursa-Wolf and Molodensky model for OLS and TLS

Table 4 presents a summary statistics on the total horizontal residuals produced when the least squares adjustment techniques were executed on the Bursa-Wolf and Molodensky-Badekas model.

Table 4: Statistics of the entire horizontal residuals

<table>
<thead>
<tr>
<th>Performance Index</th>
<th>Bursa-Wolf</th>
<th></th>
<th>Molodensky-Badekas</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>OLS (m)</td>
<td>TLS (m)</td>
<td>OLS (m)</td>
<td>TLS (m)</td>
</tr>
<tr>
<td>Max</td>
<td>1.806</td>
<td>1.738</td>
<td>1.806</td>
<td>1.808</td>
</tr>
<tr>
<td>Min</td>
<td>0.153</td>
<td>0.177</td>
<td>0.153</td>
<td>0.151</td>
</tr>
<tr>
<td>Mean</td>
<td>0.881</td>
<td>0.887</td>
<td>0.881</td>
<td>0.881</td>
</tr>
<tr>
<td>SD</td>
<td>0.392</td>
<td>0.385</td>
<td>0.392</td>
<td>0.393</td>
</tr>
</tbody>
</table>

The maximum and minimum values (Table 4) indicate the error range that could be achieved in the geodetic network of Ghana when OLS and TLS are applied to the Bursa-Wolf and Molodensky-Badekas model, respectively. It was uncovered from Table 4 that both the OLS and TLS results do not exhibit any significant distinction. This can also be seen from the mean horizontal error results (Table 4) where the values are hovering around 0.88 m. A graphical representation of the maximum (upper whisker), minimum (bottom whisker) and mean error (●) is given by Fig. 8. The SD values signify the transformation accuracy achieved by Bursa-Wolf and Molodensky-Badekas model when OLS and TLS were employed.
5. Conclusion

A wide range of least squares adjustment techniques are known and utilised in geodetic sciences for coordinate transformations between non-geocentric and geocentric datums. Prominent among them is the OLS and TLS procedures. Several studies have been conducted in different geodetic networks of countries to test the efficiency of OLS and TLS methods in coordinate transformation. Although these least squares methods are known and widely used, empirical study and their comparison are yet to be carried out in Ghana’s geodetic reference network.

In continuation of this, the present authors applied OLS and TLS to two conformal models, namely, Bursa-Wolf and Molodensky-Badekas. The obtained result from the OLS and TLS has been compared. The findings revealed that, in Ghana geodetic reference network, both the OLS and TLS produced nearly equal transformation results. This can be confirmed from the summary statistics results presented in Table 4. However, the present study results are in contrary to the reported results obtained in other jurisdictions where TLS was the better technique. The general conclusion drawn from the results of this study is that both the OLS and TLS approaches can be used for the transformation of coordinates in Ghana geodetic reference network with similar achievable accuracy.

Acknowledgment

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6. References

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